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Lesson 10.5: Elementary Series & Alternating Series

In this lesson, you will be using the four elementary power series. You are expected to know them from memory.

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{(2n-1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^n x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \cdots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}$$

$$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots + \frac{(-1)^{n-1}(x-1)^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-1)^n}{n}$$

Creating New Power Series

Examples: Using these elementary series, find a power series for each of the following functions. Show four terms and the general term.

1. $\sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots + \frac{(-1)^{n-1} x^{2(2n-1)}}{(2n-1)!} + \cdots$

2. $\cos(\sqrt{x}) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$

3. $xe^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots + \frac{x^n}{(n-1)!} + \cdots$

4. $\frac{1}{x} = 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n (x-1)^n + \cdots$

5. $\int_1^t \ln(x)\,dx = \frac{(t-1)^2}{2} - \frac{(t-1)^3}{6} + \frac{(t-1)^4}{12} - \frac{(t-1)^5}{30} + \cdots + \frac{(-1)^{n-1}(t-1)^{n+1}}{n(n+1)} + \cdots$

Alternating Series

An Alternating Series is a series whose terms alternate between positive and negative. Three of the four elementary series are alternating series.

Alternating Series Test for Convergence

Let $a_n > 0$.
The alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ & $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ converge if:
1) $a_{n+1} \le a_n \; \forall n$ after a certain n (terms never increase in absolute value)

2) $\lim_{n\to\infty} a_n = 0$

Examples: Determine the convergence or divergence.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)$

1) $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

2) $\frac{1}{n}$ is decreasing on $[1,\infty)$
$\Rightarrow a_{n+1} \le a_n \; \forall n \in [1,\infty)$ ✓

$\Rightarrow \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}\right)$ converges by AST.

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n+1}{n}\right)$

1) $\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} 1 + \cancel{\frac{1}{n}}^{\,0}$
$= 1 \neq 0$

$\Rightarrow \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{n+1}{n}\right)$ diverges by n^{th} term test.

Summary of Three Tests for Convergence:

1) n^{th} Term Test for divergence: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges. The test is inconclusive if $\lim_{n\to\infty} a_n = 0$.

2) Geometric Series Test: $|r| \ge 1 \rightarrow$ diverges / $|r| < 1 \rightarrow$ converges and $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$.

3) Alternating Series Test for Convergence (stated above).