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Lesson 10.7: Integral Test and *p*-Series

Integral Test

If the function $f(x)$ is positive, continuous, and decreasing for $x \geq 1$ and $a_n = f(n)$, then:

$\sum_{n=1}^{\infty} a_n$ & $\int_1^{\infty} f(x)\,dx$ either BOTH converge or BOTH diverge.

Note:

Examples: Use the integral test to determine the convergence or divergence of these series.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

$f(x) = \frac{1}{x^2+1}$ is positive, continuous, & decreasing on $[1,\infty)$ ✓

$\lim_{b\to\infty} \int_1^b \frac{1}{x^2+1}\,dx$

$= \lim_{b\to\infty} \arctan(x) \Big|_1^b$

$= \lim_{b\to\infty} [\arctan(b) - \arctan(1)]$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by integral test

2. $\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$

$f(x) = \frac{1}{x\ln(x)}$ is positive, continuous & decreasing on $[2,\infty)$ ✓

$\lim_{b\to\infty} \int_2^b \frac{1}{x\ln(x)}\,dx$ $\quad u = \ln(x)$, $du = \frac{1}{x}dx$

$= \lim_{b\to\infty} \int_{\ln(2)}^{\ln(b)} \frac{1}{u}\,du$

$= \lim_{b\to\infty} \ln|u| \Big|_{\ln(2)}^{\ln(b)}$

$= \lim_{b\to\infty} \ln|\overset{\infty}{\cancel{\ln(b)}}| - \ln|\ln(2)|$

$= \infty$ $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n\ln(n)}$ diverges by integral test

3. $\sum_{n=1}^{\infty} \frac{1}{n}$

$f(x) = \frac{1}{x}$ is positive, continuous & decreasing on $[1,\infty)$ ✓

$\lim_{b\to\infty} \int_1^b \frac{1}{x}\,dx$

$= \lim_{b\to\infty} \ln|x| \Big|_1^b$

$= \lim_{b\to\infty} \overset{\infty}{\cancel{\ln|b|}} - \ln|1|$

$= \infty$. $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by integral test

4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

$f(x) = \frac{1}{\sqrt{x}}$ is positive, continuous & decreasing on $[1,\infty)$ ✓

$\lim_{n\to\infty} \int_1^b \frac{1}{\sqrt{x}}\,dx$

$= \lim_{n\to\infty} \int_1^b x^{-1/2}\,dx$

$= \lim_{n\to\infty} 2\sqrt{x} \Big|_1^b$

$= \lim_{n\to\infty} \overset{\infty}{\cancel{2\sqrt{b}}} - 2$

$= \infty$ $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by integral test

5. $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$f(x) = \frac{1}{x^2}$ is positive, continuous, & decreasing on $[1,\infty)$ ✓

$\lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx$

$= \lim_{b\to\infty} \int_1^b x^{-2}\,dx$

$= \lim_{b\to\infty} -\frac{1}{x} \Big|_1^b$

$= \lim_{b\to\infty} -\overset{0}{\cancel{\frac{1}{b}}} - \left(-\frac{1}{1}\right)$

$= 1$ $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by integral test.

p-Series and Harmonic Series

If p is a positive constant, then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ is called a p-Series.

Examples 3, 4, & 5 were all examples of p-Series.

p-Series Test

If $p > 1$, then the p-series converges.

If $0 < p \leq 1$, then the p-series diverges.

Harmonic Series

The **harmonic series** is the p-Series in which $p = 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots$$

This series diverges & the name of this series may be referenced to justify divergence.

Examples:

1. $1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

$= \sum_{n=1}^{\infty} \frac{1}{n(n^{1/2})}$

$= \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

$p = \frac{3}{2} > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ converges by PST.

2. $\sum_{n=1}^{\infty} n^3 \sqrt[3]{n^{-11}} = \sum_{n=1}^{\infty} n^3 n^{-11/3} = \sum_{n=1}^{\infty} n^{9/3} n^{-11/3} = \sum_{n=1}^{\infty} n^{-2/3} = \sum_{n=1}^{\infty} \frac{1}{n^{2/3}}$

$p = \frac{2}{3} \Rightarrow 0 < p \leq 1$

$\Rightarrow \sum_{n=1}^{\infty} n^3 \sqrt[3]{n^{-11}}$ diverges by PST