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Lesson 10.8: Direct Comparison Test & Limit Comparison Test

Direct Comparison Test

Let $0 < a_n \le b_n$ for all n after a certain n.

1) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Informally:

1. If the "larger" series converges, then the "smaller" series must also converge.
2. If the "smaller" series diverges, then the "larger" series must also diverge.

Examples: Determine the convergence or divergence of the following.

1. $\sum_{n=1}^{\infty} \frac{1}{1+2^n}$ compare to: $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \rightarrow \left|\frac{1}{2}\right| = \frac{1}{2} < 1$

   $\Rightarrow \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges by GST

   $0 < \frac{1}{1+2^n} \le \frac{1}{2^n} \quad \forall n \in [1, \infty)$

   So, $\sum_{n=1}^{\infty} \frac{1}{1+2^n}$ converges by DCT.

2. $\sum_{n=3}^{\infty} \frac{1}{n-2}$ compare to $\sum_{n=3}^{\infty} \frac{1}{n} \Rightarrow$ diverges because it is the Harmonic Series.

   $0 < \frac{1}{n} < \frac{1}{n-2} \quad \forall n \in [3, \infty)$

   So, $\sum_{n=3} \frac{1}{n-2}$ diverges by DCT.

3. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ Compare to $\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ diverges because it is the Harmonic Series.

   $0 < \frac{1}{n} < \frac{1}{\sqrt{n}+1} \quad \forall n \in [3, \infty)$

   So, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+1}$ diverges by DCT.

Limit Comparison Test

If $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$, where L is finite & positive, then both $\sum_{n=1}^{\infty} a_n$ & $\sum_{n=1}^{\infty} b_n$ converge or they both diverge.

This limit comparison works well when comparing messy algebraic series to an easier *p*-Series, disregard all but the highest powers of *n* in the numerator and denominator.

Examples: Determine the convergence or divergence of the following.

1. $\sum_{n=1}^{\infty} \frac{1}{3n^2-4}$

$\frac{1}{n^2} > 0$ & $\frac{1}{3n^2-4} > 0 \quad \forall n \in [2,\infty)$

Compare to $\frac{1}{n^2}$ $\quad p = 2 > 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by PST

$$\lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{3n^2-4}{1} = \lim_{n\to\infty} \frac{3n^2-4}{n^2} = 3 > 0$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{3n^2-4}$ converges by LCT.

2. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+7}$

$\frac{1}{\sqrt{n}} > 0$ & $\frac{\sqrt{n}}{n+7} > 0 \quad \forall n \in [1,\infty)$

Compare to $\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$ $\quad p = \frac{1}{2} < 1 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges by PST

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{n+7}{\sqrt{n}} = \lim_{n\to\infty} \frac{n+7}{n} = 1 > 0$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n+7}$ diverges by LCT

3. $\sum_{n=1}^{\infty} \frac{n}{4n^3+n^2+5}$

$\frac{1}{n^2} > 0$ & $\frac{n}{4n^3+n^2+5} > 0 \quad \forall n \in [1,\infty)$

compare to: $\frac{1}{n^2}$ $\quad p = 2 > 0 \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by PST

$$\lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{4n^3+n^2+5}{n} = \lim_{n\to\infty} \frac{4n^3+n^2+5}{n^3} = 4 > 0$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n}{4n^3+n^2+5}$ converges by LCT