Entry #: 55

Lesson 10.9: Ratio Test *useful for series involving factorials or exponentials.

1) $\sum a_n$ converges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$.

2) $\sum a_n$ diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$

3) The ratio test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

Examples: Determine the convergence or divergence.

1. $\sum_{n=0}^{\infty}\frac{n!}{3^n}$

$$\lim_{n\to\infty}\left|\frac{(n+1)!}{3^{n+1}}\cdot\frac{3^n}{n!}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(n+1)\cancel{n!}}{3\cdot\cancel{3^n}}\cdot\frac{\cancel{3^n}}{\cancel{n!}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{n+1}{3}\right| = \infty$$
$\Rightarrow \sum_{n=0}^{\infty}\frac{n!}{3^n}$ diverges by Ratio Test.

2. $\sum_{n=1}^{\infty}\frac{3^{n+1}}{4^n n^2} = \sum_{n=1}^{\infty}\frac{3\cdot 3^n}{4^n n^2}$

$$\lim_{n\to\infty}\left|\frac{3\cdot 3^{n+1}}{4^{n+1}(n+1)^2}\cdot\frac{4^n n^2}{3\cdot 3^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{3\cdot\cancel{3}\cdot\cancel{3^n}}{4\cdot\cancel{4^n}(n+1)^2}\cdot\frac{\cancel{4^n}n^2}{\cancel{3}\cdot\cancel{3^n}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{3n^2}{4(n+1)^2}\right|$$
$$= \frac{3}{4} < 1$$
$\Rightarrow \sum_{n=1}^{\infty}\frac{3^{n+1}}{4^n n^2}$ converges by RT

3. $\sum_{n=2}^{\infty}\frac{(-1)^n\sqrt{n}}{n-1}$ alternating ✓ decreasing on $[2,\infty)$ ✓

$$\lim_{n\to\infty}\frac{\sqrt{n}}{n-1}\cdot\frac{1/n}{1/n} = \lim_{n\to\infty}\frac{\cancel{\frac{1}{\sqrt{n}}}^{\,0}}{1-\cancel{1/n}^{\,0}}$$
$$= 0 \checkmark$$

$\Rightarrow \sum_{n=2}^{\infty}\frac{(-1)^n\sqrt{n}}{n-1}$ converges by AST

4. $\sum_{n=1}^{\infty}\frac{(2n+1)!!}{3^n(2n-1)n!} = \sum_{n=1}^{\infty}\frac{(2n+1)(2n-1)(2n-3)\cdots5\cdot3\cdot1}{3^n(2n-1)n!}$

$$\lim_{n\to\infty}\left|\frac{(2n+3)!!}{3^{n+1}(2n+1)(n+1)!}\cdot\frac{3^n(2n-1)n!}{(2n+1)!!}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(2n+3)\cancel{(2n+1)!!}}{3\cdot\cancel{3^n}(2n+1)(n+1)\cancel{n!}}\cdot\frac{\cancel{3^n}(2n-1)\cancel{n!}}{\cancel{(2n+1)!!}}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(2n+3)(2n-1)}{3(2n+1)(n+1)}\right| = \frac{4}{6} = \frac{2}{3} < 1 \Rightarrow \sum_{n=1}^{\infty}\frac{(2n+1)!!}{3^n(2n-1)n!}$$ converges by RT

Handling Mixed Problems

1. Does the n^{th} term approach zero?
   If not, the series diverges. (n^{th} term test)
2. Is the series a special type?
   ex: Geometric (r^n), p-series ($\frac{1}{n^p}$), or alternating?
3. Can you use the Integral Test or Ratio Test?
   ↑ can be integrated — ↑ factorials or exponentials
4. Can you compare to a special type using DCT or LCT?
   ↑ compare to larger/smaller series — ↑ messy algebraic

Mixed Examples:

1. $\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$

$\lim_{n\to\infty} \frac{n-1}{2n+1} = \frac{1}{2} \neq 0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n-1}{2n+1}$ diverges by nth term test.

2. $\sum_{n=1}^{\infty} n^2 e^{-n^3} = \sum_{n=1}^{\infty} \frac{n^2}{e^{n^3}}$

$\lim_{n\to\infty} \left| \frac{(n+1)^2}{e^{(n+1)^3}} \cdot \frac{e^{n^3}}{n^2} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{\cancel{e^{n^3}} \cdot e^{3n^2} \cdot e^{3n} \cdot e} \cdot \frac{\cancel{e^{n^3}}}{n^2} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2 (e^{3n^2+3n+1})} \right| = 0 < 1$

$\Rightarrow \sum_{n=1}^{\infty} n^2 e^{-n^3}$ converges by RT

3. $\sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^n$

$\left|\frac{e}{3}\right| < 1$

$\Rightarrow \sum_{n=1}^{\infty} \left(\frac{e}{3}\right)^n$ converges by GST.

4. $\sum_{n=1}^{\infty} \frac{1}{4n+5}$ $f(x) = \frac{1}{4n+5}$ is continuous, positive & decreasing on $[1,\infty)$.

$\lim_{b\to\infty} \int_1^b \frac{1}{4x+5}\,dx$

$= \lim_{b\to\infty} \frac{1}{4} \ln|4x+5| \Big|_1^b$

$= \lim_{b\to\infty} \frac{1}{4}\ln|4\overset{\infty}{\cancel{b}}+5| - \frac{1}{4}\ln|9|$

$= \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{4n+5}$ diverges by IT.

5. $\sum_{n=1}^{\infty} (-1)^n \frac{3}{n^2}$ alternating ✓ decreasing on $[1,\infty)$ ✓

$\lim_{n\to\infty} \frac{3}{n^2} = 0$ ✓

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{3}{n^2}$ converges by AST.

6. $\frac{1}{10} + \frac{1\cdot 2}{10^2} + \frac{1\cdot 2\cdot 3}{10^3} + \frac{1\cdot 2\cdot 3\cdot 4}{10^4} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{10^5} + \cdots = \sum_{n=1}^{\infty} \frac{n!}{10^n}$

$\lim_{n\to\infty} \left| \frac{(n+1)!}{10^{n+1}} \cdot \frac{10^n}{n!} \right|$

$= \lim_{n\to\infty} \left| \frac{(n+1)\cancel{n!}}{10\cdot \cancel{10^n}} \cdot \frac{\cancel{10^n}}{\cancel{n!}} \right|$

$= \lim_{n\to\infty} \left| \frac{n+1}{10} \right| = \infty$

$\Rightarrow \sum_{n=1}^{\infty} \frac{n!}{10^n}$ diverges by RT