Lesson #1: Sequences & Summation Notation

The Definition of a Sequence

A sequence is an ordered list of numbers.

For example, 2, 4, 6, 8, 10, ... is a sequence.

Each number within a sequence is called a term.

a_n stands for the n^{th} term in the sequence.

$a_1 \; a_2 \; a_3 \; a_4 \; a_5$

Example: Answer the questions below given the sequence 1, 3, 5, 7 ,9, …

1. What is the 3rd term? 5
2. Find a_6. $a_6 = 11$
3. Find a_n. $a_n = 2n-1$
4. Find a_{25}. $a_{25} = 2(25)-1 = 50-1 = 49$

Finding Terms of a Sequence

Examples: Find the first 5 terms and the 100th term of the sequence defined by each formula.

1. $a_n = 3n - 4$

$a_1 = 3(1)-4 = -1$
$a_2 = 3(2)-4 = 2$
$a_3 = 3(3)-4 = 5$
$a_4 = 3(4)-4 = 8$
$a_5 = 3(5)-4 = 11$

$a_{100} = 3(100)-4 = 296$

2. $a_n = n^2 - 2$

$a_1 = (1)^2-2 = -1$
$a_2 = (2)^2-2 = 2$
$a_3 = (3)^2-2 = 7$
$a_4 = (4)^2-2 = 14$
$a_5 = (5)^2-2 = 23$

$a_{100} = (100)^2-2 = 10{,}000-2 = 9{,}998$

3. $a_n = \frac{n+1}{n}$

$a_1 = \frac{(1)+1}{1} = 2$
$a_2 = \frac{(2)+1}{2} = \frac{3}{2}$
$a_3 = \frac{(3)+1}{(3)} = \frac{4}{3}$
$a_4 = \frac{(4)+1}{4} = \frac{5}{4}$
$a_5 = \frac{(5)+1}{5} = \frac{6}{5}$

$a_{100} = \frac{100+1}{100} = \frac{101}{100}$

4. $a_n = \frac{(-1)^n}{2^n}$

$a_1 = \frac{(-1)^1}{2^1} = -\frac{1}{2}$
$a_2 = \frac{(-1)^2}{2^2} = \frac{1}{4}$
$a_3 = \frac{(-1)^3}{2^3} = -\frac{1}{8}$
$a_4 = \frac{(-1)^4}{2^4} = \frac{1}{16}$
$a_5 = \frac{(-1)^5}{2^5} = -\frac{1}{32}$

$a_{100} = \frac{(-1)^{100}}{2^{100}} = \frac{1}{2^{100}}$

FINDING THE NTH TERM OF A *sequence*

Examples: Find the n^th^ term of a sequence whose first several terms are given.

$a_1 \ a_2 \ a_3 \ a_4$

1. $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \dots$

   $a_n = \frac{2n-1}{2n}$

$a_1 \ a_2 \ a_3 \ a_4 \ a_5$

2. $-2, 4, -8, 16, -32, \dots$

   $a_n = (-2)^n$

$a_1 \ a_2 \ a_3 \ a_4$

3. $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \dots$

   $a_n = \left(-\frac{1}{3}\right)^{n-1}$

4. $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \dots$

   $a_n = (n)^{(-1)^{n-1}}$

RECURSIVELY DEFINED *sequence*

Some sequences are not defined as simply as the examples we have seen so far.

Sometimes, the terms of a sequence depend on the value of some or all the previous terms.

A sequence defined in this way is called recursive.

If we have the term a_n, the term a_{n-1} is the term right before it.

$\dots, a_{n-2}, a_{n-1}, a_n, a_{n+1}, a_{n+2}, \dots$

Examples:

1. If $a_1 = 2$ and $a_n = 2(a_{n-1}) - 2$, find the first five terms of the sequences.

   $a_2 = 2(a_1) - 2$ $\quad$ $a_3 = 2(a_2) - 2$

   $a_2 = 2(2) - 2$ $\quad$ $a_3 = 2(2) - 2$

   $a_2 = 2$ $\quad$ $a_3 = 2$

   $2, 2, 2, 2, 2, \dots$

2. If $a_1 = -8$ and $a_n = \frac{a_{n-1}}{2}$, find the first five terms of the sequences.

   $a_2 = \frac{a_1}{2} = \frac{-8}{2} = -4$ $\quad$ $a_4 = \frac{a_3}{2} = \frac{-2}{2} = -1$

   $a_3 = \frac{a_2}{2} = \frac{-4}{2} = -2$ $\quad$ $a_5 = \frac{a_4}{2} = \frac{-1}{2}$

   $-8, -4, -2, -1, -\frac{1}{2}, \dots$

3. If $a_1 = a_2 = a_3 = 1$ and $a_n = a_{n-1} + a_{n-2} + a_{n-3}$, find the first five terms of the sequences.

   $a_4 = a_3 + a_2 + a_1$ $\quad$ $a_5 = a_4 + a_3 + a_2$

   $a_4 = 1 + 1 + 1 = 3$ $\quad$ $a_5 = 3 + 1 + 1 = 5$

   $1, 1, 1, 3, 5, \dots$

THE PARTIAL SUMS OF A *sequence*	We can add up specific terms of a sequence. S_n tells you to add up the first n terms of the sequence. **Examples:** 1. Given the sequence defined by $a_n = \frac{1}{2^n}$, find the first 3 partial sums. $a_n = \frac{1}{2^n}$ $a_1, a_2, a_3, \cdots$ $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ $S_1 = \frac{1}{2}$ $S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$ $S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$ 2. Find the first four partial sums of the sequence defined by $a_n = \frac{1}{n} - \frac{1}{n+1}$. $a_n = \frac{1}{n} - \frac{1}{n+1}$ $a_1, a_2, a_3, a_4, \cdots$ $(1-\frac{1}{2}), (\frac{1}{2}-\frac{1}{3}), (\frac{1}{3}-\frac{1}{4}), (\frac{1}{4}-\frac{1}{5}), \ldots$ $S_1 = (1-\frac{1}{2}) = \frac{1}{2}$ $S_2 = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) = 1-\frac{1}{3} = \frac{2}{3}$ $S_3 = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) = 1-\frac{1}{4} = \frac{3}{4}$ $S_4 = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + (\frac{1}{3}-\frac{1}{4}) + (\frac{1}{4}-\frac{1}{5}) = 1-\frac{1}{5} = \frac{4}{5}$
SIGMA *notation*	We can also write the sum of the first n terms of a sequence using sigma notation: $\sum_{j=1}^{n} a_j = a_1 + a_2 + a_3 + \cdots + a_n$
	Examples: Find each sum. 1. $\sum_{k=1}^{4} k^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2$ $= 1 + 4 + 9 + 16$ $= 5 + 25$ $= 30$ 2. $\sum_{j=3}^{5} \frac{1}{j} = \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$ $= \frac{20}{60} + \frac{15}{60} + \frac{12}{60}$ $= \frac{47}{60}$ 3. $\sum_{i=4}^{8} i = 4 + 5 + 6 + 7 + 8$ $= 30$ 4. $\sum_{k=1}^{5} 3 = 3 + 3 + 3 + 3 + 3$ $= 15$

SIGMA *notation*

Examples: Write each sum using sigma notation.

1. $1^2 + 2^2 + 3^2 + 4^2 + 5^2$

$a_n = n^2$

$$\sum_{n=1}^{5} n^2$$

2. $\frac{\sqrt{1}}{1^2} + \frac{\sqrt{2}}{2^2} + \frac{\sqrt{3}}{3^2} + \cdots \frac{\sqrt{n}}{n^2}$

$a_n = \frac{\sqrt{n}}{n^2}$

$$\sum_{n=1}^{n} \frac{\sqrt{n}}{n^2}$$

3. $\sqrt{2} + \sqrt{3} + \sqrt{4} + \sqrt{5} + \cdots + \sqrt{62}$

$a_n = \sqrt{n}$

$$\sum_{n=2}^{62} \sqrt{n}$$

4. $1 - 2x + 3x^2 - 4x^3 + 5x^4 + \cdots - 100x^{99}$

$a_n = (-1)^{n-1} n x^{n-1} = n(-x)^{n-1}$

$$\sum_{n=1}^{100} n(-x)^{n-1}$$

PROPERTIES OF *sums*

Let $a_1, a_2, a_3, a_4, \ldots$ and $b_1, b_2, b_3, b_4, \ldots$ be sequences. Then, the following properties of sums hold:

$$\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

$$\sum_{k=1}^{n} c a_k = c\left(\sum_{k=1}^{n} a_k\right)$$