Lesson 2.7: Limits at Infinity

<table>
<tr><td rowspan="5">LIMITS AT infinity</td><td colspan="2">So far, we have only considered limits as x approaches a finite value c.

In this lesson, we will consider limits where x approaches ∞ or $-\infty$.</td></tr>
<tr><td>Limit</td><td>Meaning</td></tr>
<tr><td>$\lim_{x\to\infty} f(x) = L$</td><td>As x gets larger in the positive direction, f(x) converges to L.</td></tr>
<tr><td>$\lim_{x\to-\infty} f(x) = L$</td><td>As x gets larger (in absolute value) in the negative direction, f(x) converges to L.</td></tr>
<tr><td colspan="2">In either case, the line $y = L$ is called a horizontal asymptote.

Infinite limits describe the asymptotic behavior of a function, which is the behavior of the graph as we move out to the right or the left. In other words, the end behavior of the function.

Examples: Discuss the asymptotic behavior of the function given the limits below.

1. $\lim_{x\to\infty} g(x) = 7$ and $\lim_{x\to-\infty} g(x) = 3$
g(x) has a horizontal asymptote at y=7 to the right and y=3 to the left.

2. $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$
f(x) does not have a horizontal asymptote.

3. $\lim_{x\to\infty} 2^x = \infty$ and $\lim_{x\to-\infty} 2^x = 0$
The function 2^x has a horizontal asymptote at y=0 to the left.

4. $\lim_{x\to\infty} \sin(x)$ DNE and $\lim_{x\to-\infty} \sin(x)$ DNE
The function sin(x) oscillates, so it does not ever converge to a value at its end behavior.</td></tr>
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THEOREM 1

For all $n > 0$,

$$\lim_{x\to\infty} x^n = \infty \qquad \lim_{x\to\infty} \frac{1}{x^n} = 0$$

If n is a whole number,

$$\lim_{x\to-\infty} x^n = \begin{cases} \infty & \textit{if } n \textit{ is even} \\ -\infty & \textit{if } n \textit{ is odd} \end{cases} \qquad \text{and} \qquad \lim_{x\to-\infty} x^{-n} = \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

LIMITS AT INFINITY OF *rational functions*

We can take a limit at infinity of a rational function to determine the function's horizontal asymptote.

Examples: Determine if the functions below have a horizontal asymptote by taking limits at infinity.

1. $f(x) = \frac{20x^2-3x}{3x^5-4x^2+5}$

$$\lim_{x\to\infty} \frac{20x^2-3x}{3x^5-4x^2+5} = \lim_{x\to\infty} \frac{20x^2}{3x^5} = 0$$

$$\lim_{x\to-\infty} \frac{20x^2-3x}{3x^5-4x^2+5} = 0$$

H.A. @ $y=0$.

2. $f(x) = \frac{3x^4-7x+9}{7x^4-4}$

$$\lim_{x\to\infty} \frac{3x^4-7x+9}{7x^4-4} = \lim_{x\to\infty} \frac{3x^4}{7x^4} = \frac{3}{7}$$

$$\lim_{x\to-\infty} \frac{3x^4-7x+9}{7x^4-4} = \lim_{x\to-\infty} \frac{3x^4}{7x^4} = \frac{3}{7}$$

H.A. @ $y=\frac{3}{7}$

3. $f(x) = \frac{3x^8-7x+9}{7x^4-4}$

$$\lim_{x\to\infty} \frac{3x^8-7x+9}{7x^4-4} = \lim_{x\to\infty} \frac{3x^8}{7x^4} = \lim_{x\to\infty} \frac{3}{7}x^4 = \infty$$

$$\lim_{x\to-\infty} \frac{3x^8-7x+9}{7x^4+4} = \lim_{x\to-\infty} \frac{3x^8}{7x^4} = \lim_{x\to-\infty} \frac{3}{7}x^4 = \infty$$

No horizontal asymptote

4. $f(x) = \frac{3x^{\frac{7}{2}}+7x^{-\frac{1}{2}}}{x^2-x^{\frac{1}{2}}}$

$$\lim_{x\to\infty} \frac{3x^{7/2}+7x^{-1/2}}{x^2-x^{1/2}} = \lim_{x\to\infty} \frac{3x^{7/2}}{x^2} = \lim_{x\to\infty} 3x^{3/2} = \infty$$

$$\lim_{x\to-\infty} \frac{3x^{7/2}+7x^{-1/2}}{x^2-x^{1/2}} = \lim_{x\to-\infty} \frac{3x^{7/2}}{x^2} = \lim_{x\to-\infty} 3x^{3/2}$$

DNE (out of domain)

No horizontal asymptote.

5. $f(x) = \frac{x^2}{\sqrt{x^3+1}}$

$$\lim_{x\to\infty} \frac{x^2}{\sqrt{x^3+1}} = \lim_{x\to\infty} \frac{x^2}{\sqrt{x^3}} = \lim_{x\to\infty} \frac{x^2}{x^{3/2}} = \lim_{x\to\infty} x^{1/2} = \infty$$

$$\lim_{x\to-\infty} \frac{x^2}{\sqrt{x^3+1}} = \lim_{x\to-\infty} \frac{x^2}{\sqrt{x^3}} = \lim_{x\to-\infty} x^{1/2}$$

DNE

No horizontal asymptotes.

6. $f(x) = \frac{12x+25}{\sqrt{16x^2+100x+500}}$

$$\lim_{x\to\infty} \frac{12x+25}{\sqrt{16x^2+100x+500}} = \lim_{x\to\infty} \frac{12x}{\sqrt{16x^2}} = \lim_{x\to\infty} \frac{12x}{4|x|} = \lim_{x\to\infty} \frac{12x}{4x} = 3$$

$$\lim_{x\to-\infty} \frac{12x+25}{\sqrt{16x^2+100x+500}} = \lim_{x\to-\infty} \frac{12x}{\sqrt{16x^2}} = \lim_{x\to-\infty} \frac{12x}{4|x|} = \lim_{x\to-\infty} \frac{12x}{4(-x)} = -3$$

H.A. @ $y=3$ to the right and $y=-3$ to the left.