LESSON #2: ARITHMETIC SEQUENCES

THE DEFINITION OF AN ARITHMETIC *sequence*

An arithmetic sequence is created by adding a constant value to the previous term to get the next term.

For example: $\underbrace{3, 6, 9, 12, 15, \ldots}_{+3\ +3\ +3\ +3}$ is an arithmetic sequence where 3 is being added to each term to get the next term.

In general, an arithmetic sequence is of the form:

$$a, a+d, a+2d, a+3d, \ldots$$

The number a is the first term and the number d is the common difference.

We can write the n^{th} term of an arithmetic sequence as:

$$a_n = a_1 + (n-1)d$$

IDENTIFYING ARITHMETIC *sequences*

Examples: Identify whether the sequences below are arithmetic or not. If they are arithmetic, find the common difference and n^{th} term.

1. $\underbrace{1, 3, 5, 7, 9}_{+2\ +2\ +2\ +2}, \ldots$

   common difference: 2

   $a_n = 1 + (n-1)(2)$

   $a_n = 1 + 2n - 2$

   $a_n = 2n - 1$

2. $\underbrace{-5, -8, -11, -14}_{-3\ -3\ -3}, \ldots$

   common difference: -3

   $a_n = -5 + (n-1)(-3)$

   $a_n = -5 - 3n + 3$

   $a_n = -3n - 2$

3. $\underbrace{1, 1, 2, 3, 5}_{+0\ +1\ +1\ +2}, \ldots$

   no common difference

   ⇒ not arithmetic

4. $\underbrace{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}}_{+\frac{1}{2}\ +\frac{1}{2}\ +\frac{1}{2}\ +\frac{1}{2}}, \ldots$

   common difference: $\frac{1}{2}$

   $a_n = \frac{1}{2} + (n-1)(\frac{1}{2})$

   $a_n = \frac{1}{2} + \frac{1}{2}n - \frac{1}{2}$ $\quad a_n = \frac{1}{2}n$

5. $\underbrace{0, -\frac{1}{5}, -\frac{2}{5}, -\frac{3}{5}}_{-\frac{1}{5}\ -\frac{1}{5}\ -\frac{1}{5}}, \ldots$

   common difference: $-\frac{1}{5}$

   $a_n = 0 + (n-1)(-\frac{1}{5})$

   $a_n = -\frac{1}{5}n + \frac{1}{5}$

6. $\underbrace{4, 8, 16, 32}_{+4\ +8\ +16}, \ldots$

   no common difference

   ⇒ not arithmetic

   *is there another pattern though? *geometric →next class!

FINDING TERMS OF AN ARITHMETIC *sequence*

Examples: Find the first 5 terms and the 100th term for each of the arithmetic sequences below.

1. $\underbrace{12, 11}_{-1}, \ldots$

$a_1 = 12$
$a_2 = 11$
$a_3 = 10$
$a_4 = 9$
$a_5 = 8$

$a_n = 12 + (n-1)(-1)$
$a_n = 12 - n + 1$
$a_n = -n + 13$

$a_{100} = -100 + 13$
$= -87$

2. $\overbrace{3, 10}^{+7}, \ldots$

$a_1 = 3$
$a_2 = 10$
$a_3 = 17$
$a_4 = 24$
$a_5 = 31$

$a_n = 3 + (n-1)(7)$
$a_n = 3 + 7n - 7$
$a_n = 7n - 4$

$a_{100} = 7(100) - 4$
$= 696$

3. $\underbrace{0, \frac{1}{3}}_{+1/3}, \ldots$

$a_1 = 0$
$a_2 = 1/3$
$a_3 = 2/3$
$a_4 = 1$
$a_5 = 4/3$

$a_n = 0 + (n-1)(\frac{1}{3})$
$a_n = \frac{1}{3}n - \frac{1}{3}$

$a_{100} = \frac{1}{3}(100) - \frac{1}{3}$
$= 33$

4. $\underbrace{-2, -\frac{5}{2}}_{-1/2}, \ldots$

$a_1 = -2$
$a_2 = -5/2$
$a_3 = -3$
$a_4 = -7/2$
$a_5 = -4$

$a_n = -2 + (n-1)(-\frac{1}{2})$
$a_n = -2 - \frac{1}{2}n + \frac{1}{2}$
$a_n = -\frac{1}{2}n - 1$

5. $a_1 = 8$ and $a_{10} = 44$

$a_n = 8 + (n-1)d$
$a_{10} = 8 + (10-1)d$
$44 = 8 + 9d$
$-8 \quad -8$
$\frac{36}{9} = \frac{9d}{9}$
$d = 4$

$a_1 = 8$
$a_2 = 12$
$a_3 = 16$
$a_4 = 20$
$a_5 = 24$

$a_n = 8 + (n-1)(4)$
$a_n = 8 + 4n - 4$
$a_n = 4n + 4$
$a_{100} = 4(100) + 4$
$= 404$

6. $a_{11} = 52$ and $a_{19} = 92$

$52 = a_1 + (11-1)d$
$52 = a_1 + 10d$
$92 = a_1 + (19-1)d$
$92 = a_1 + 18d$
$\frac{40}{8} = \frac{8d}{8}$
$d = 5$

$52 = a_1 + 10(5)$
$52 = a_1 + 50$
$a_1 = 2$

$a_n = 2 + (n-1)(5)$
$a_n = 5n - 3$

$a_1 = 2$
$a_2 = 7$
$a_3 = 12$
$a_4 = 17$
$a_5 = 22$

$a_{100} = 5(100) - 3$
$= 497$

FINDING PARTIAL SUMS OF AN ARITHMETIC *sequence*

To find S_n (the nth partial sum) of an arithmetic sequence we can use the following formula:

$$S_n = n\left(\frac{a_1 + a_n}{2}\right)$$

Examples: Find the indicated partial sum in each problem below.

1. Find the sum of the first 40 terms of the sequence: $\overbrace{3, 7}^{+4}, \overbrace{11}^{+4}, \overbrace{15}^{+4}, \ldots$

$a_n = 3 + (n-1)(4)$
$a_n = 3 + 4n - 4$
$a_n = 4n - 1$
$a_{40} = 4(40) - 1$
$= 159$

$S_{40} = 40\left(\frac{3 + 159}{2}\right)$
$= 3{,}240$

2. Find the sum of the first 50 odd numbers. $1, 3, 5, 7, \cdots$

$a_n = 1 + (n-1)(2)$
$a_n = 2n - 1$
$a_{50} = 2(50) - 1$
$= 99$

$S_{50} = 50\left(\frac{1 + 99}{2}\right)$
$= 2{,}500$

3. Find the sum of the first 100 natural numbers.

$S_{100} = 100\left(\frac{1 + 100}{2}\right)$
$= 100\left(\frac{101}{2}\right)$
$= 100(50.5)$
$= 5{,}050$

4. Find the sum of the first 50 terms of the sequence: $\underbrace{6, 0}_{-6}, \underbrace{-6}_{-6}, \underbrace{-12}_{-6}, \ldots$

$a_n = 6 + (n-1)(-6)$
$a_n = -6n + 12$
$a_{50} = -6(50) + 12$
$= -288$

$S_{50} = 50\left(\frac{6 + (-288)}{2}\right)$
$= -7{,}050$