Entry #: 11

Lesson 3.1: Implicit Differentiation

All the derivatives you have done to this point have been explicit equations. These are equations that are explicitly written in terms of x.

In this lesson, you will be working with implicit equations where the relationship between x and y is only implied.

For example, $x^2 + y^2 = 1, xy + y^2 = 3$, and $xy = 1$ are all implicit equations.

It is possible to differentiate implicit equations using implicit differentiation.

Procedure

1) Differentiate both sides w.r.t. x.
   * Remember the y' "hook on factor" for any term involving y
2) Collect all y' terms on one side of the equation
3) Factor out y'.
4) Divide to solve for y'.

Warm-Up Examples: Differentiate

1. $y = x$

$$\frac{dy}{dx} = \frac{dx}{dx}$$
$$\frac{dy}{dx} = 1$$

2. $y = x^2$

$$\frac{dy}{dx} = \frac{dx^2}{dx}$$
$$\frac{dy}{dx} = 2x\frac{dx}{dx}$$
$$\frac{dy}{dx} = 2x$$

3. $y = (2x-1)^2$

$$\frac{dy}{dx} = 2(2x-1)(2)\frac{dx}{dx}$$
$$\frac{dy}{dx} = 4(2x-1)$$
$$\frac{dy}{dx} = 8x-4$$

4. $y = (f(x))^2$

$$\frac{dy}{dx} = 2(f(x))(f'(x))$$

Examples:

1. Given $x^2 - 2y^3 + 3x = 6$, find y'.

$$\frac{d}{dx}(x^2 - 2y^3 + 3x) = \frac{d}{dx}(6)$$
$$2x - 6y^2 \cdot y' + 3 = 0$$
$$-2x \qquad -3 \qquad -2x-3$$
$$\frac{-6y^2 \cdot y'}{-6y^2} = \frac{-2x-3}{-6y^2}$$
$$y' = \frac{2x+3}{6y^2}$$

2. Find the slope of the tangent line to and normal to the graph of $x^2 + 4y^2 = 25$ at (3,2).

$\frac{d}{dx}(x^2+4y^2) = \frac{d}{dx}(25)$

$2x + 8y \cdot y' = 0$

$-2x \qquad -2x$

$\frac{8y \cdot y'}{8y} = \frac{-2x}{8y} \Rightarrow y' = -\frac{x}{4y}$

$m = -\frac{3}{4(2)} = -\frac{3}{8}$

$m_{norm} = \frac{8}{3}$

3. Given $x^3 - 2xy + y^3 = 5x$, find $\frac{dy}{dx}$ and evaluate at the point (1,2).

$\frac{d}{dx}(x^3 - 2xy + y^3) = \frac{d}{dx}(5x)$

$3x^2 - (2xy' + 2y) + 3y^2 \cdot y' = 5$

$3x^2 - 2y - 2xy' + 3y^2 y' = 5$

$-3x^2 + 2y \qquad -3x^2 + 2y$

$-2xy' + 3y^2 y' = 5 - 3x^2 + 2y$

$\frac{y'(-2x+3y^2)}{(-2x+3y^2)} = \frac{5-3x^2+2y}{(-2x+3y^2)}$

$y' = \frac{-3x^2+2y+5}{3y^2-2x}$

$y'|_{(1,2)} = \frac{-3(1)^2+2(2)+5}{3(2)^2-2(2)} = \frac{6}{8} = \frac{3}{4}$

4. Given $x^2 + y^2 = 3$, find $\frac{dx}{dy^2}$ (or x'') with respect to y.

$\frac{d}{dy}(x^2+y^2) = \frac{d}{dy}(3)$

$2x \cdot x' + 2y = 0$

$-2y \quad -2y$

$\frac{2x \cdot x'}{2x} = \frac{-2y}{2x}$

$x' = -\frac{y}{x}$

$x'' = \frac{d}{dy}\left(-\frac{y}{x}\right)$

$x'' = \frac{-x - (-y)x'}{(x)^2}$

$x'' = \frac{-x + y\left(-\frac{y}{x}\right)}{x^2} = \frac{1}{x^2}\left(-x - \frac{y^2}{x}\right)$

$= -\frac{1}{x} - \frac{y^2}{x^3}$

5. Given $\cot(y) = x - y$ find $\frac{dy}{dx}$.

$\frac{d}{dx}\cot(y) = \frac{d}{dx}(x-y)$

$(-\csc^2(y))y' = 1 - y'$

$+y' \qquad +y'$

$(-\csc^2(y))y' + y' = 1$

$\frac{y'(-\csc^2(y)+1)}{(-\csc^2(y)+1)} = \frac{1}{(-\csc^2(y)+1)}$

$y' = \frac{1}{-\csc^2(y)+1}$