ENTRY #32

Lesson #3: Geometric Sequences

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<td>THE DEFINITION OF A GEOMETRIC sequence</td>
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A geometric sequence is created by multiplying a constant value to the previous term to get the next term.

For example: $\underbrace{1, 3, 9, 27, \ldots}_{\times 3\ \times 3\ \times 3}$ is a geometric sequence where 3 is being multiplied to each term to get the next term.

In general, a geometric sequence is of the form:
$a, ar, ar^2, ar^3, ar^4, \ldots$

The number a is the first term and the number r is the common ratio.

We can write the n^{th} term of a geometric sequence as:
$a_n = a_1 r^{n-1}$
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<td>IDENTIFYING GEOMETRIC sequences</td>
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Examples: Identify whether the sequences below are geometric or not. If they are geometric, find the common ratio and nth term.

1. $\underbrace{2, 4, 8, 16, 32}_{\times 2\ \times 2\ \times 2\ \times 2}, \ldots$
common ratio: 2
$a_n = 2(2)^{n-1}$

2. $\underbrace{3, -12, 48, -192}_{\times -4\ \times -4\ \times -4}, \ldots$
common ratio: -4
$a_n = 3(-4)^{n-1}$

3. $2, 2, 4, 6, 10, \ldots$
no common ratio
$\Rightarrow$ not geometric

4. $\underbrace{10, 5, \frac{5}{2}, \frac{5}{4}}_{\times \frac{1}{2}\ \times \frac{1}{2}\ \times \frac{1}{2}}, \ldots$
common ratio: 1/2
$a_n = 10(\frac{1}{2})^{n-1}$

5. $\underbrace{5, 3, 1, -1}_{-2\ -2\ -2}, \ldots$
* not geometric
this is actually an arithmetic sequence with a common ratio of -2.

6. $\underbrace{-18, 6, -2, \frac{2}{3}}_{\times -\frac{1}{3}\ \times -\frac{1}{3}\ \times -\frac{1}{3}}, \ldots$
common ratio: $-\frac{1}{3}$
$a_n = -18(-\frac{1}{3})^{n-1}$
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FINDING TERMS OF A GEOMETRIC *sequence*

Examples:

1. Find the eighth term of the geometric sequence 5, 15, 45, ...

$a_n = 5(3)^{n-1}$

$a_8 = 5(3)^{8-1} = 5(3)^7 = 10{,}935$

2. Find the tenth term of the geometric sequence $-27, 9, -3, \ldots$

$a_n = -27(-\frac{1}{3})^{n-1}$

$a_{10} = -27(-\frac{1}{3})^{10-1} = -27(-\frac{1}{3})^9 = \frac{1}{729}$

3. The third term of a geometric sequence is $\frac{63}{4}$ and the sixth term is $\frac{1701}{32}$. Find the fifth term.

$a_3 = \frac{63}{4}$ $\qquad \frac{63}{4} = a_1 r^2$ $\qquad \frac{a_1 r^5}{a_1 r^2} = \frac{\frac{1701}{32}}{\frac{63}{4}}$

$a_6 = \frac{1701}{32}$ $\qquad \frac{1701}{32} = a_1 r^5$ $\qquad \sqrt[3]{r^3} = \sqrt[3]{\frac{27}{8}}$

$r = \frac{3}{2}$

$a_5 = \frac{1701}{32} \div \frac{3}{2} = \frac{1701}{32} \cdot \frac{2}{3} = \boxed{\frac{567}{16}}$

PARTIAL SUMS OF A GEOMETRIC *sequence*

For a geometric sequence of the form $a_n = ar^{n-1}$, the n^{th} partial sum (S_n) is given by:

$$S_n = a_1\left(\frac{1-r^n}{1-r}\right)$$

Examples: Find the partial sum in the problems below.

1. Find the sum of the first $\overset{n=5}{\underline{\text{five}}}$ terms of the sequence: $\underbrace{1, 0.7}_{\times 0.7}, \underbrace{0.7, 0.49}_{\times 0.7}, \underbrace{0.49, 0.343}_{\times 0.7}, \ldots$ $\quad a_1 = 1$

$S_5 = 1\left(\frac{1-0.7^5}{1-0.7}\right)$

$= 2.7731$

2. Find the sum $\sum_{k=1}^{5} 7\left(-\frac{2}{3}\right)^{k-1}$ $\quad r = -\frac{2}{3}$, $a_1 = 7$

$\sum_{k=1}^{5} 7(-\frac{2}{3})^k = 7\left(\frac{1-(-2/3)^5}{1-(-2/3)}\right)$

$= 7\left(\frac{55}{81}\right)$

$= \frac{385}{81}$

INFINITE series

An infinite series is a sum in which we are adding up an infinite number of terms :

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + \cdots$$

If the sum of infinite series is equal to a finite value, we say the series converges .

If the series does not converge, we say the series diverges .

Example: Show the infinite series $\sum_{k=1}^{\infty} \frac{1}{2^k}$ converges.

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

$$\left.\begin{aligned} S_1 &= \tfrac{1}{2} \\ S_2 &= \tfrac{1}{2} + \tfrac{1}{4} = \tfrac{3}{4} \\ S_3 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} = \tfrac{7}{8} \\ S_4 &= \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} = \tfrac{15}{16} \end{aligned}\right\} \text{getting closer and closer to } 1$$

$$S_n = \frac{2^n - 1}{2^n} = 1 - \frac{1}{2^n} \leftarrow \text{as } n \to \infty,\ \frac{1}{2^n} \to 0$$

Therefore, $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$

INFINITE GEOMETRIC series

If $|r| < 1$, then the infinite geometric series $a + ar + ar^2 + ar^3 + ar^4 + \cdots$ converges and has the sum:

$$S = \frac{a_1 \leftarrow \text{first term}}{1 - r \leftarrow \text{common ratio}}$$

Examples: Determine if the infinite geometric series below converge or diverges. Explain your reasoning. If the series converges, evaluate the sum.

1. $\underbrace{2 + \frac{2}{5}}_{\times\frac{1}{5}} \underbrace{+ \frac{2}{25}}_{\times\frac{1}{5}} \underbrace{+ \frac{2}{125}}_{\times\frac{1}{5}} + \cdots = \frac{5}{2}$

   $r = \frac{1}{5}$ $\quad |\frac{1}{5}| < 1$ ✓

   ⇒ This geometric series converges

   $S = \frac{2}{1 - 1/5} = \frac{2}{4/5} = \frac{10}{4} = \boxed{\frac{5}{2}}$

2. $\underbrace{1 + \frac{5}{4}}_{\times 5/4} \underbrace{+ \left(\frac{5}{4}\right)^2}_{\times 5/4} \underbrace{+ \left(\frac{5}{4}\right)^3}_{\times 5/4} + \cdots$

   $r = \frac{5}{4}$ $\quad |\frac{5}{4}| \not< 1$

   ⇒ This geometric series diverges.

3. $\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1} = \frac{8}{3}$

   $r = -\frac{1}{2}$ $\quad |-\frac{1}{2}| < 1$

   ⇒ $\sum_{k=1}^{\infty} 4\left(-\frac{1}{2}\right)^{k-1}$ converges

   $S = \frac{4}{1 - (-1/2)} = \frac{4}{3/2} = \frac{8}{3}$

4. $\sum_{k=1}^{\infty} \left(\frac{6}{5}\right)^{k-1}$

   $r = \frac{6}{5}$ $\quad |\frac{6}{5}| \not< 1$

   ⇒ $\sum_{k=1}^{\infty} \left(\frac{6}{5}\right)^{k-1}$ diverges