LESSON 4.1: LINEAR APPROXIMATIONS & APPLICATIONS

LINEAR *approximations*

In some situations, we are interested in determining the effect of a "small change".

For example, we may want to know what a slight change in angle will impact the distance of a basketball shot.

If we have a function $f(x)$, we can represent the "small change" as:

$$\Delta f = f(a+\Delta x) - f(a)$$

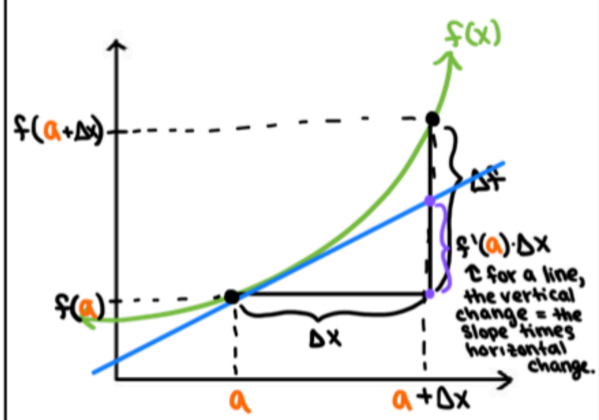
*where Δx is small.

Linear approximation or sometimes referred to as <u>tangent</u> line approximation is calculated as follows:

If f is differentiable at $x = a$ and Δx is small, then

$$\Delta f \approx f'(a)\Delta x$$

$$\text{where } \Delta f = f(a+\Delta x) - f(a)$$



Examples:

1. Use the linear approximation to estimate $\frac{1}{10.2} - \frac{1}{10}$. How accurate is your estimate?

$f(x) = \frac{1}{x}$ where $a = 10$ & $\Delta x = 0.2$

$$\Delta f = f(10.2) - f(10) \approx -0.00196$$

$$\Delta f \approx f'(10)(0.2) = -\frac{1}{10^2}(0.2) = -0.002$$

Error $\approx |-0.00196 - (-0.002)| = 0.00004 < 10^{-4}$

2. How much larger is $\sqrt[3]{8.1}$ than $\sqrt[3]{8} = 2$?

$f(x) = \sqrt[3]{x}$ where $a = 8$ & $\Delta x = 0.1$

$f'(x) = \frac{1}{3}x^{-2/3}$

$$\Delta f \approx f'(8)(0.1) = \frac{1}{3}(8)^{-2/3}(0.1) = \frac{1}{3}\left(\frac{1}{4}\right)(0.1) = \frac{1}{12}(0.1) = \frac{1}{120} \approx 0.0083$$

Therefore $\sqrt[3]{8.1}$ is larger than $\sqrt[3]{8}$ by about $\frac{1}{120}$.

LINEAR *approximations*	**Examples:**

3. A thin metal cable has length $L = 12\ cm$ when the temperature is $T = 21°C$. Estimate the change in length when T rises to 24°C, assuming that $\frac{dL}{dT} = kL$ where $k = 1.7 \times 10^{-5}°C$ (k is called the coefficient of thermal expansion).

$\Delta T = 3°C \qquad L = 12$

$$\Delta L = \left(\frac{dL}{dt}\right)\Delta T$$

$$\left.\frac{dL}{dt}\right|_{L=12} = kL = (1.7\times10^{-5})(12) \approx 2\times10^{-4}\ cm/°C$$

$$\Delta L \approx (2\times10^{-4})(3) = 6\times10^{-4}\ cm$$

4. The Bonzo Pizza Company claims that its pizzas are circular with diameter 50 cm.

   a. What is the area of the pizza?

$r = \frac{d}{2} \qquad A = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi}{4}d^2$

$$A = \pi\left(\frac{50}{2}\right)^2 = \pi(25)^2 \approx 1963.495$$

   b. Estimate the quantity of pizza lost or gained if the diameter is off by at most 1.2 cm.

$\Delta d = 1.2$

$$\Delta A = A'(50)(1.2) \qquad A' = \frac{\pi}{2}d$$

$$= \frac{50\pi}{2}(1.2)$$

$$\approx 94.248$$

$\pm 94.248\ cm^2$ will be gained / lost

LINEARIZATION	To approximate $f(x)$ by its linearization we use the following: $$f(x) \approx L(x) = f'(a)(x-a) + f(a)$$
	Examples: 1. Compute the linearization of $f(x) = \sqrt{x}$ at $a = 1$. $f'(x) = \frac{1}{2}x^{-1/2}$ $f(x) \approx f'(1)(x-1) + f(1)$ $= \frac{1}{2}(x-1) + 1$ $= \frac{1}{2}x - \frac{1}{2} + \frac{2}{2}$ $= \frac{1}{2}x + \frac{1}{2}$ ← this can be used to approximate function values close to 1. 2. Estimate $\tan\left(\frac{\pi}{4} + 0.02\right)$ and compute the percentage error. $f(x) = \tan(x)$ $a = \frac{\pi}{4}$ $f(x) \approx f'(\pi/4)(x - \pi/4) + f(\pi/4)$ $= 2(x - \pi/4) + 1$ $x = \frac{\pi}{4} + 0.02$ $\tan(\frac{\pi}{4} + 0.02) \approx 2((0.02 + \pi/4 - \pi/4)) + 1$ $= 2(0.02) + 1$ $= 1.04$ $\tan(\pi/4 + 0.02) \approx 1.0408$ % error: $\left\lvert \frac{1.0408 - 1.04}{1.0408} \right\rvert \cdot 100\%$ $\approx 0.08\%$