LESSON 4.6: APPLIED OPTIMIZATION

APPLIED *optimization*

In many real-world situations, it is necessary to find to find an optimal length or amount of something.

For example, if we are on a budget for materials while building a fence, we can determine how to maximize the enclosed area to stay within our budget.

These application problems will involve finding a maximum or minimum of a function on a particular interval. This requires finding critical values and testing endpoints to find the minimum or maximum.

Examples:

1. A piece of wire of length L is bent into the shape of a rectangle as in the image below. Which dimensions produce the rectangle of the maximum area?

$0 < x < \frac{L}{2}$

$A = (x)\left(\frac{L}{2} - x\right)$

$A = \frac{L}{2}x - x^2$

$A' = \frac{L}{2} - 2x$

$\frac{L}{2} - 2x = 0$

$\quad +2x \quad +2x$

$\frac{\frac{L}{2}}{2} = \frac{2x}{2}$

$\frac{L}{4} = x$

A': $+$ | $-$ at $\frac{L}{4}$

A has a relative maximum at $\frac{L}{4}$.

$x = \frac{L}{4}$

$\frac{L}{2} - \frac{L}{4} = \frac{L}{4}$

$\boxed{\text{The dimensions of the rectangle that will maximize the area would be } \frac{L}{4} \text{ by } \frac{L}{4}.}$

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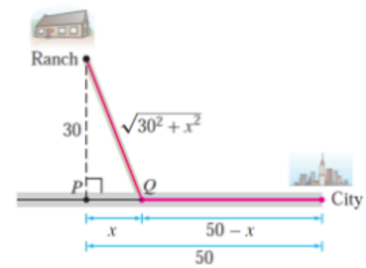
2. Your task is to build a road joining a ranch to a highway that enables drivers to reach the city in the shortest time. How should this be done if the speed limit is 60 km/h on the road and 110 km/h on the highway? The perpendicular distance from the ranch to the highway is 30 km, and the city is 50 km down the highway.



$t = \frac{d}{v}$

$T(x) = \left(\text{time to go from the ranch to Q}\right) + \left(\text{time to go from Q to the city}\right)$

$= \frac{\sqrt{30^2+x^2}}{60} + \frac{50-x}{110}$

$T'(x) = \left(\frac{1}{60}\right)\left(\frac{1}{2}\right)(30^2+x^2)^{-1/2}(2x) + \left(-\frac{1}{110}\right)$

$= \frac{x}{60\sqrt{30^2+x^2}} - \frac{1}{110}$

$\frac{x}{60\sqrt{30^2+x^2}} - \frac{1}{110} = 0$

$\frac{x}{60\sqrt{30^2+x^2}} = \frac{1}{110}$

$\frac{110x}{60} = \frac{60\sqrt{30^2+x^2}}{60}$

$\left(\frac{11}{6}x\right)^2 = \left(\sqrt{30^2+x^2}\right)^2$

$\frac{121}{36}x^2 = 30^2 + x^2$

$-x^2 \qquad -x^2$

$\frac{85}{36}x^2 = 900$

$\sqrt{x^2} = \sqrt{(900)\left(\frac{36}{85}\right)}$

$x \approx 19.524$

The travel time is minimized if the road joins the highway about 19.524 kilometers from point Q.

3. All units in a 30-unit apartment building are rented out when the monthly rent is set at r = \$1000/month. A survey reveals that one unit becomes vacant with each \$40 increase in rent. Suppose that each occupied unit costs \$120/month in maintenance. Which rent r maximizes monthly profit?

$\text{Profit} = \left(\#\text{ of units occupied}\right)\left(\text{monthly rent} - \text{maitenence cost}\right)$

Let r = monthly rent

$P(r) = \left(30 - \frac{1}{40}(r - 1000)\right)(r - 120)$

$= \left(30 - \frac{1}{40} + \frac{1000}{40}\right)(r-120)$

$= \left(-\frac{1}{40}r + 5\right)(r-120)$

$P(1000) = 26{,}400$

$P(1160) = 27{,}040$

$P(2200) = 0$

$P'(r) = \left(-\frac{1}{40}\right)(r-120) + \left(-\frac{1}{40}r + 55\right)$

$= -\frac{1}{40}r + 3 - \frac{1}{40}r + 55$

$= -\frac{1}{20}r + 58$

$58 - \frac{1}{20}r = 0$

$-58 \qquad -58$

$(-20)\left(-\frac{1}{20}r\right) = (-58)(-20)$

$r = 1160$

$P''(r) = -\frac{1}{20} < 0$

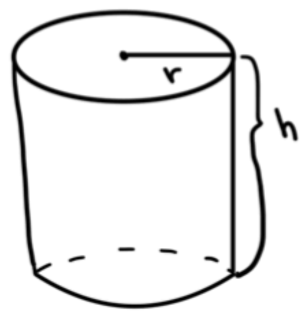
$P''(1160) < 0$

$\Rightarrow$ there is a relative max at $r = 1160$

The rent that would maximize profit is \$1,160 per month.

APPLIED optimization

4. Design a cylindrical can of volume 900 cm^3 so that it uses the least amount of metal. In other words, minimize the surface area of the can (including its top and bottom).



$$SA = 2\pi r^2 + 2\pi r h$$

$$SA = 2\pi r^2 + 2\pi r\left(\frac{900}{\pi r^2}\right)$$

$$SA = 2\pi r^2 + \frac{1800}{r}$$

$$SA' = 4\pi r - \frac{1800}{r^2}$$

$$V = \pi r^2 h$$

$$\frac{900}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2}$$

$$h = \frac{900}{\pi r^2}$$

$$h = \frac{900}{\pi\left(\sqrt[3]{\frac{450}{\pi}}\right)^2}$$

$$h \approx 10.464$$

$$4\pi r - \frac{1800}{r^2} = 0$$

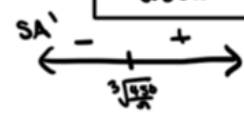
$$4\pi r = \frac{1800}{r^2}$$

$$\sqrt[3]{r^3} = \sqrt[3]{\frac{1800}{4\pi}}$$

$$r = \sqrt[3]{\frac{1800}{4\pi}}$$

$$r = \sqrt[3]{\frac{450}{\pi}}$$

$r \approx 5.232$ ← there is a relative min where $r = \sqrt[3]{\frac{450}{\pi}}$



$\boxed{\text{To minimize the surface area, the radius needs to be about 5.232 cm and the height needs to be about 10.464 cm.}}$

5. Is it possible to design a cylinder of volume 900 cm^3 with the largest possible surface area?

$$SA = 2\pi r^2 + \frac{1800}{r}$$

The SA function has no maximum because as $r \to 0$, $SA \to \infty$ and as $r \to \infty$, $SA \to \infty$. This means, a cylinder has a large surface area if it is either very fat and short or very tall and skinny.