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Lesson 6.3: Definite Integrals & The Fundamental Theorem of Calculus

Definite Integrals

A definite integral is written with upper and lower limits attached to an integration expression.

The value of a definite integral $\int_a^b f(x)\,dx$ may be thought of as "signed" area from the lower limit a (usually a left side boundary) to the upper limit b (usually a right side boundary), and between the curve of $f(x)$ and the x-axis.

The value of definite integrals may be positive, negative, or zero.

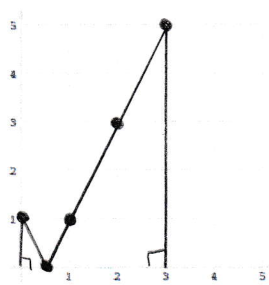
Unlike the previous integration process which introduced an indefinite integral representing a family of curves, a definite integral represents a **number value.**

Examples:

1. Use your calculator to evaluate the following integrals.

   a. $\int_{-3}^{1}(x^3 - 6x)\,dx = 4$ b. $\int_{-\sqrt{6}}^{\sqrt{6}}(x^3 - 6x)\,dx = 0$ c. $\int_{-5}^{5}|x^3 - 6x|\,dx \approx 198.500$

2. Use the idea of "signed area" to evaluate $\int_0^3 |2x - 1|\,dx$ without using a calculator.



$$\int_0^3 |2x-1|\,dx = \tfrac{1}{2}(1)(\tfrac{1}{2}) + \tfrac{1}{2}(\tfrac{5}{2})(5)$$
$$= \frac{1}{4} + \frac{25}{4}$$
$$= \frac{26}{4}$$
$$= \frac{13}{2}$$

3. Set up a definite integral which could be used to find the area of the region bounded by the graph of $y = 2x^2 - 3x + 2$, the x-axis, and the vertical lines $x = 0$ and $x = 2$.

$$\int_0^2 2x^2 - 3x + 2\,dx$$

The Fundamental Theorem of Calculus

> If f' is continuous on $[a, b]$, then $\int_a^b f'(x)dx = f(x)\Big|_a^b = f(b) - f(a)$

Examples: Evaluate using the Fundamental Theorem of Calculus.

1. $\int_0^4 (2\sqrt{y} + 1)dy$

$= \int_0^4 2y^{1/2} + 1\, dy$

$= \frac{4}{3} y^{3/2} + y \Big|_0^4$

$= \left[\frac{4}{3}(4)^{3/2} + 4\right] - \left[\frac{4}{3}(0)^{3/2} + 0\right] \approx 14.667$

2. $\int_0^1 (4t + 1)^5\, dt$

$= \frac{1}{4}\int_0^1 4(4t+1)^5\, dt$

$= \frac{1}{24}(4t+1)^6 \Big|_0^1$

$= \frac{1}{24}(4(1)+1)^6 - \frac{1}{24}(4(0)+1)^6 = 651$

Let $u = 2x-1$
$du = 2dx$
$\frac{1}{2}du = dx$

3. $\int_1^5 \frac{x}{\sqrt{2x-1}}\, dx$

$= \frac{1}{2}\int_1^9 \frac{\left(\frac{u+1}{2}\right)}{\sqrt{u}}\, du$

$= \frac{1}{4}\int_1^9 u^{1/2} + u^{-1/2}\, du$

$= \frac{1}{6}u^{3/2} + \frac{1}{2}u^{1/2} \Big|_1^9$

$= \left[\frac{1}{6}(9)^{3/2} + \frac{1}{2}(9)^{1/2}\right] - \left[\frac{1}{6}(1)^{3/2} + \frac{1}{2}(1)^{1/2}\right]$

$= \frac{27}{6} + \frac{3}{2} - \frac{1}{6} - \frac{1}{2}$

$= \frac{32}{6} = \frac{16}{3}$

4. $\int_0^{\frac{\pi}{2}} \cos(2x)\, dx$

$= \frac{1}{2}\int_0^{\pi/2} 2\cos(2x)dx$

$= \frac{1}{2}\sin(2x)\Big|_0^{\pi/2}$

$= \frac{1}{2}\sin\left(2\left(\frac{\pi}{2}\right)\right) - \frac{1}{2}\sin(0)$

$= \frac{1}{2}\sin(\pi) - \frac{1}{2}\sin(0) = 0$

Start Plus Accumulation Method (SPAM)

Since $\int_a^b f'(x)\, dx = f(b) - f(a)$, it follows that $f(b) = f(a) + \int_a^b f'(x)dx$. This means a function value at an endpoint can be found as a starting value plus a definite integral.

Examples:

1. If $f'(x) = 3x^2 + 3$ and $f(0) = 4$, find $f(2)$ without a calculator.

$\int_0^2 3x^2 + 3\, dx = f(2) - f(0)$

$x^3 + 3x \Big|_0^2 = f(2) - 4$

$[2^3 + 3(2)] - [0^3 + 3(0)] = f(2) - 4$

$8 + 6 = f(2) - 4$

$14 = f(2) - 4$

$+4 \qquad +4$

$f(2) = 18$

2. If an object's velocity is $v(t) = 2^t$ and $s(2) = 8$ find $s(3)$.

$\int_2^3 2^t\, dt = s(3) - s(2)$

$\frac{2^t}{\ln(2)}\Big|_2^3 = s(3) - 8$

$\frac{2^3}{\ln(2)} - \frac{2^2}{\ln(2)} = s(3) - 8$

$s(3) = \frac{8}{\ln(2)} - \frac{4}{\ln(2)} + 8$

$s(3) \approx 13.771$