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Lesson 6.7: Integration by Parts

Integration by parts is a method of integration used mainly for products of algebraic and transcendental functions (such as $\int xe^x dx$) or products of two transcendental functions (such as $\int e^x \sin(x)\,dx$).

Development of the formula for integration by parts:

If u and v are both functions of x, then $\frac{d}{dx}(uv) = u'v + uv'$

Formula for Integration by Parts:

$$\int uv'\,dx = uv - \int vu'\,dx \quad \text{OR} \quad \int u\,dv = uv - \int v\,du$$

Strategy for Integration by Parts:

- Let u be the part whose derivative is "simpler" (or at least no more complicated) than u itself.
- Let dv be the more complicated part (or the part which can easily be integrated).
- Also, remember that typically have only two choices, if one doesn't work try the other.

Examples:

1. $\int xe^x dx$

Let $u = x$
Let $du = dx$
Let $dv = e^x dx$
Let $v = e^x$

$$\int xe^x dx = xe^x - \int e^x dx$$
$$= xe^x - e^x + C$$

2. $\int x\sin(3x)dx$

Let $u = x$
Let $du = dx$
Let $dv = \sin(3x)$
Let $v = -\frac{1}{3}\cos(3x)$

$$\int x\sin(3x)dx = uv - \int v\,du$$
$$= -\frac{1}{3}x\cos(3x) + \int \frac{1}{3}\cos(3x)dx$$
$$= -\frac{1}{3}x\cos(3x) + \frac{1}{9}\sin(3x) + C$$

3. $\int \arcsin(x)\,dx$

Let $u = \arcsin(x)$
Let $du = \frac{1}{\sqrt{1-x^2}}$
Let $dv = dx$
Let $v = x$

Let $\ell = 1-x^2$
$d\ell = -2x\,dx$

$$\int \arcsin(x)\,dx = uv - \int v\,du$$
$$= x\arcsin(x) - \int x\left(\frac{1}{\sqrt{1-x^2}}\right)dx$$
$$= x\arcsin(x) - \int \frac{x}{\sqrt{1-x^2}}dx$$
$$= x\arcsin(x) + \frac{1}{2}\int \frac{1}{\sqrt{\ell}}d\ell$$
$$= x\arcsin(x) + \frac{1}{2}\int \ell^{-1/2}d\ell$$
$$= x\arcsin(x) + \frac{1}{2}(2)\ell^{1/2} + C$$
$$= x\arcsin(x) + \sqrt{1-x^2} + C$$

4. $\int x^2 \sin(2x)\,dx$ $= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}\int(\cos(2x))(2x)\,dx$

$= -\frac{1}{2}x^2\cos(2x) + \int x\cos(2x)\,dx$

Let $u = x^2$

$du = 2x\,dx$

$dv = \sin(2x)$

$v = -\frac{1}{2}\cos(2x)$

$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \frac{1}{2}\int\sin(2x)\,dx$

$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$

Let $l = x$

$dl = dx$

$dm = \cos(2x)$

$m = \frac{1}{2}\sin(2x)$

5. $\int_1^e x^2 \ln(x)\,dx$ $= \frac{1}{3}x^3\ln(x)\Big|_1^e - \int\frac{1}{3}x^3\left(\frac{1}{x}\right)dx$

Let $u = \ln(x)$

$du = \frac{1}{x}\,dx$

$dv = x^2\,dx$

$v = \frac{1}{3}x^3$

$= \frac{1}{3}x^3\ln(x)\Big|_1^e - \frac{1}{3}\int x^2\,dx$

$= \left[\frac{e^3}{3}\right] - \frac{1}{9}x^3\Big|_1^e$

$= \left[\frac{e^3}{3}\right] - \left[\frac{e^3}{9} - \frac{1}{9}\right]$

$= \frac{3e^3}{9} - \frac{e^3}{9} + \frac{1}{9}$

$= \frac{2e^3}{9} + \frac{1}{9} = \frac{2e^3+1}{9}$

6. Complete the square to find $\int \frac{1}{x^2+4x+8}\,dx$.

$x^2 + 4x + 8 = 0$

$\underline{\quad -8 \quad -8}$

$x^2 + 4x + \underline{4} = -8 + \underline{4}$

$(x+2)^2 = -4$

$(x+2)^2 + 4 = 0$

$\int \frac{1}{(x+2)^2+4}\,dx$

Let $u = x+2$

$du = dx$

$a = 2$

$= \int \frac{1}{(x+2)^2+(2)^2}\,dx$

$= \frac{1}{2}\arctan\left(\frac{x+2}{2}\right) + C$