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Lesson 6.8: Partial Fractions, Mixed Integration

We have often simplified an expression like $\frac{1}{x-4}-\frac{1}{x-3}$ by getting a common denominator and combining the two fractions into one.

By a reverse process, we can sometimes split a single fraction in two to make integration easier.

Examples:

1. $\int \frac{1}{x^2-7x+12}dx = \int \frac{1}{(x-4)(x-3)}dx$

$= \int \frac{1}{x-4} - \frac{1}{x-3}\,dx$

$= \ln|x-4| - \ln|x-3| + C$

$\frac{1}{(x-4)(x-3)} = \frac{A}{x-4} + \frac{B}{x-3}$

$\frac{1}{(x-4)(x-3)} = \frac{A(x-3)}{(x-4)(x-3)} + \frac{B(x-4)}{(x-4)(x-3)}$

$1 = A(x-3) + B(x-4)$

Let $x=3$ Let $x=4$

$1=-B$ $1=A$

$B=-1$

2. $\int \frac{5x-3}{x^2-2x-3}dx = \int \frac{5x-3}{(x-3)(x+1)}dx$

$= \int \frac{3}{x-3} + \frac{2}{x+1}\,dx$

$= 3\ln|x-3| + 2\ln|x+1| + C$

$\frac{5x-3}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$\frac{5x-3}{(x-3)(x+1)} = \frac{A(x+1)}{(x-3)(x+1)} + \frac{B(x-3)}{(x-3)(x+1)}$

$5x-3 = A(x+1) + B(x-3)$

Let $x=-1$.   Let $x=3$

$-8=-4B$ $12=4A$

$B=2$ $A=3$

3. $\int \frac{2x-2}{x^2-2x-3}dx = \int \frac{2x-2}{(x-3)(x+1)}dx$

$= \int \frac{1}{x-3} + \frac{1}{x+1}\,dx$

$= \ln|x-3| + \ln|x+1| + C$

$= \ln|(x-3)(x+1)| + C$

$= \ln|x^2-2x-3| + C$

$\frac{2x-2}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}$

$\frac{2x-2}{(x-3)(x+1)} = \frac{A(x+1)}{(x-3)(x+1)} + \frac{B(x-3)}{(x-3)(x+1)}$

$2x-2 = A(x+1) + B(x-3)$

Let $x=-1$ Let $x=3$

$-4=-4B$ $4=4A$

$B=1$ $A=1$

4. $\int \frac{x^3-x+2}{x^2+x-2}dx$

$= \int x-1 + \frac{2x}{x^2+x-2}dx$

$= \int x-1 + \frac{2x}{(x+2)(x-1)}dx$

$= \int x-1 + \frac{\frac{4}{3}}{x+2} + \frac{\frac{2}{3}}{x-1}dx$

$= \frac{1}{2}x^2 - x + \frac{4}{3}\ln|x+2| + \frac{2}{3}\ln|x-1| + C$

$$\begin{array}{r} x-1+\frac{2x}{x^2+x-2} \\ x^2+x-2\,\big)\,\overline{x^3-x+2} \\ -(x^3+x^2-2x) \\ -x^2+x+2 \\ -(-x^2-x+2) \\ 2x \end{array}$$

$\frac{2x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$

$\frac{2x}{(x+2)(x-1)} = \frac{A(x-1)}{(x+2)(x-1)} + \frac{B(x+2)}{(x+2)(x-1)}$

$2x = A(x-1) + B(x+2)$

Let $x=1$ Let $x=-2$

$2=3B$ $-4=-3A$

$B=\frac{2}{3}$ $A=\frac{4}{3}$

5. Integrate these four "look-alike" integrals. Although they have similar appearances, they will require you to use three completely different integration formulas.

a. $\int \frac{dx}{1+x^2} = \int \frac{1}{(1)^2+(x)^2} dx$

$= \arctan(x) + C$

b. $\int \frac{dx}{1+x} = \ln|1+x| + C$

c. $\int \frac{x\,dx}{1+x^2} = \frac{1}{2}\int \frac{2x}{1+x^2} dx$

$= \frac{1}{2}\ln|1+x^2| + C$

d. $\int \frac{dx}{(1+x)^2}$

Let $u = 1+x$

$du = dx$

$= \int \frac{1}{u^2} du$

$= \int u^{-2} du$

$= -\frac{1}{u} + C$

$= -\frac{1}{1+x} + C$

6. Integrate the following two "look-alikes".

a. $\int \frac{dx}{x\ln(x)} = \int \frac{1}{u} du$

Let $u = \ln(x)$

$du = \frac{1}{x} dx$

$= \ln|u| + C$

$= \ln|\ln(x)| + C$

b. $\int \frac{\ln(x)}{x} dx = \int u\, du$

Let $u = \ln(x)$

$du = \frac{1}{x} dx$

$= \frac{1}{2} u^2 + C$

$= \frac{1}{2} \ln^2(x) + C$