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Lesson 6.9: Improper Integrals

An integral is called improper if:

1. one or both limits of integration are infinite.
2. the function has an infinite discontinuity (vertical asymptote) at or between the limits of integration.

Examples: Use the conditions above to explain why each of the following integrals is improper.

1. $\int_1^{\infty} \frac{1}{x}\,dx$

   The upper limit of integration is infinite.

3. $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$

   Both upper & lower limits of integration are infinite.

2. $\int_1^{5} \frac{1}{\sqrt{x-1}}\,dx$

   $\frac{1}{\sqrt{x-1}}$ has a vertical asymptote at $x=1$. $1 \in [1,5]$

4. $\int_{-2}^{2} \frac{1}{(x+1)^2}\,dx$

   $\frac{1}{(x+1)^2}$ has a vertical asymptote @ $x=-1$. $-1 \in [2,-2]$

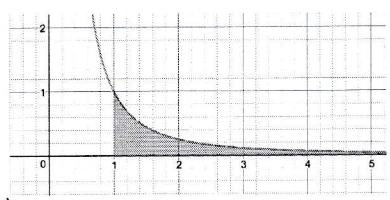
Examples: Evaluate the following improper integrals. Identify those which **diverge**.

1. $\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t x^{-2}\,dx$

$= \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t$

$= \lim_{t\to\infty} \left[-\cancel{\tfrac{1}{t}}^{0} + 1\right]$

$= 1 \leftarrow$ converges.



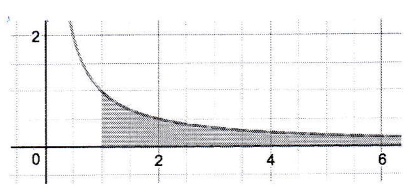
2. $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$

$= \lim_{t\to\infty} \ln|x| \Big|_1^t$

$= \lim_{t\to\infty} \left[\cancel{\ln|t|}^{\infty} - \ln|1|\right]$

$= \infty$

$\Rightarrow \int_1^{\infty} \frac{1}{x}\,dx$ diverges



3. $\int_0^\infty \cos(x)\,dx = \lim_{t\to\infty} \int_0^t \cos(x)dx = \lim_{t\to\infty} \sin(x)\Big|_0^t$

$= \lim_{t\to\infty} [\sin(t) - \sin(0)]$

$= \lim_{t\to\infty} \sin(t)$ DNE

$\Rightarrow \int_0^\infty \cos(x)dx$ diverges

4. $\int_1^\infty xe^{-x}\,dx = \lim_{t\to\infty} \int_1^t xe^{-x}dx = \lim_{t\to\infty} -xe^{-x}\Big|_1^t + \int_1^t e^{-x}dx$

Let $u = x$
$du = dx$
$dv = e^{-x}dx$
$v = -e^{-x}$

$= \lim_{t\to\infty} \left[-te^{-t} + \frac{1}{e}\right] - e^{-x}\Big|_1^t$

$= \lim_{t\to\infty} \left[\cancel{-te^{-t}}^{\,0} + \frac{1}{e}\right] - \left[\cancel{e^{-t}}^{\,0} - \frac{1}{e}\right]$

$= \frac{1}{e} + \frac{1}{e}$

$= \frac{2}{e}$

5. $\int_{-1}^2 \frac{dx}{x^3} = \lim_{a\to 0^-} \int_{-1}^a x^{-3}dx + \lim_{b\to 0^+} \int_b^2 x^{-3}dx$

$= \lim_{a\to 0^-} \left[-\frac{1}{2}x^{-2}\Big|_{-1}^a\right] + \lim_{b\to 0^+} \left[-\frac{1}{2}x^{-2}\Big|_b^2\right]$

$= \lim_{a\to 0^-} \left[\cancel{-\frac{1}{2a^2}}^{\,-\infty} + \frac{1}{2}\right] + \lim_{b\to 0^+} \left[-\frac{1}{2}\left(\frac{1}{4}\right) + \cancel{\frac{1}{2}(b)^{-2}}^{\,\infty}\right] \leftarrow$ DNE

$\Rightarrow \int_{-1}^2 \frac{1}{x^3}dx$ diverges