Entry #: 42

Lesson 9.1: Parametric Equations

Examples:

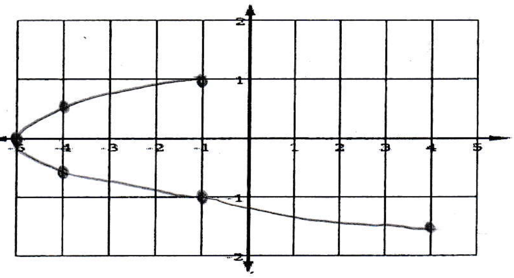
1. Plot points to sketch the curve described by the parametric equations. Mark the orientation on the curve.

$x = t^2 - 5$

$y = \frac{t}{2}$

$-3 \le t \le 2$

t	-3	-2	-1	0	1	2
x	4	-1	-4	-5	-4	-1
y	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1



2. Change the following to rectangular form by eliminating the parameter. Then graph.

$x = \frac{1}{\sqrt{t+1}}$ and $y = \frac{t}{t+1}$, $t > -1$.

$(x)^2 = \left(\frac{1}{\sqrt{t+1}}\right)^2$

$x^2 = \frac{1}{t+1}$

$\frac{x^2(t+1)}{x^2} = \frac{1}{x^2}$

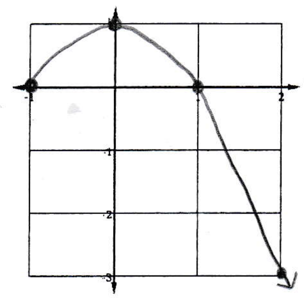
$t+1 = \frac{1}{x^2} \Rightarrow t = \frac{1}{x^2} - 1$

$y = \frac{\left(\frac{1}{x^2}-1\right)}{\left(\frac{1}{x^2}-1\right)+1}$

$y = \frac{\frac{1}{x^2}-1}{\frac{1}{x^2}}$

$y = \left(\frac{1}{x^2}-1\right)(x^2)$

$y = 1 - x^2$

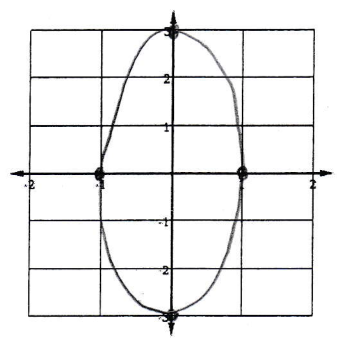


3. Eliminate the parameter to sketch the curve. $x = \cos(\theta)$ and $y = 3\sin(\theta)$, $0 \le \theta \le 2\pi$

$x = \cos(\theta)$

$\frac{y}{3} = \frac{3\sin(\theta)}{3} \Rightarrow \frac{y}{3} = \sin(\theta)$

$\sin^2(\theta) + \cos^2(\theta) = 1 \Rightarrow x^2 + \left(\frac{y}{3}\right)^2 = 1$



Even though equations are given in terms of a parameter, it is possible to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ by differentiating then dividing:

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)} \qquad \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{dt}\right)}$$

1. If $x = \cos(t)$ and $y = 3\sin(t)$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$\frac{dx}{dt} = -\sin(t)$ $\quad \frac{dy}{dt} = 3\cos(t)$

$\frac{dy}{dx} = \frac{3\cos(t)}{-\sin(t)}$

$= -3\cot(t)$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(-3\cot(t))}{-\sin(t)} = \frac{-3\csc^2(t)}{\sin(t)}$

2. Find the slope and concavity of $x = \sqrt{t}$ and $y = \frac{1}{4}t^2 - 1$, $t \geq 0$ at the point (2,3).

$(2)^2 = (\sqrt{t})^2$

$t = 4$

$\frac{dy}{dt} = \frac{1}{2}t \quad \frac{dx}{dt} = \frac{1}{2}t^{-1/2}$

$\frac{dy}{dx} = \frac{\frac{1}{2}t}{\frac{1}{2}t^{-1/2}} = t^{3/2}$

$\frac{dy}{dx}(4) = (4)^{3/2} = 8$ slope @ (2,3)

$\frac{d^2y}{dx^2} = \frac{\frac{3}{2}t^{1/2}}{\frac{1}{2}t^{-1/2}}$

$= \frac{3}{2}t^{1/2}(2t^{1/2})$

$= 3t$ positive when $t=4$

$\Rightarrow$ concave up at (2,3)

3. Write an equation of a tangent line to the curve defined by $x = t - 1$ and $y = \frac{1}{t} + 1$ at the point when $t = 1$.

$\frac{dy}{dt} = -t^{-2} \quad \frac{dx}{dt} = 1$

$\frac{dy}{dx} = -\frac{1}{t^2}$

$\frac{dy}{dx}(1) = -1$ slope when $t=1$

$x = (1) - 1 = 0 \quad y = \frac{1}{1} + 1 = 2 \quad (0,2)$

$y - 2 = -(x)$

$\boxed{y = -x + 2}$

Arc Length

If a curve is smooth and does not intersect itself, the length of the arc is given by:

$$\text{arc length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example: Using the parametric equations from example 3 above, find the arc length on the interval $1 \leq t \leq 3$.

$\int_1^3 \sqrt{(1)^2 + (-t^{-2})^2}\, dt = \int_1^3 \sqrt{1 + t^{-4}}\, dt \approx 2.147$